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Short Communication

Nonlinear dynamic behaviors of clamped laminated shallow shells with one-to-one internal resonance

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Abstract

This paper investigates one-to-one internal resonance of laminated shallow shells with rigidly clamped edges. It is assumed that the natural frequencies ω_2 and ω_3 of two asymmetric (second and third) vibration modes have the relationship $\omega_2 \approx \omega_3$. The displacements are expressed by using eigenvectors for linear vibration modes calculated by the Ritz method. Applying Galerkin's procedure to the equation of motion, nonlinear differential equations are derived. By considering the first vibration mode in addition to the two asymmetric vibration modes, quadratic nonlinear terms expressing the interaction between the asymmetric and the first modes appear in the differential equations. Shooting method is used to obtain the steady-state response when the driving frequency Ω is near ω_2. The dynamic characteristics of the shells with the internal resonance are discussed.

1. Introduction

When thin structures are subjected to dynamic loads and vibrate with finite amplitudes which are the order of their thickness, nonlinear responses such as superharmonic, subharmonic, internal and combination resonances may occur. Since they cannot be predicted by linear models, nonlinear models taking into account geometric nonlinearities are essential for accurate prediction of those behaviors. Therefore, many researchers have applied geometric nonlinear theories and studied nonlinear dynamic characteristics of plates and shells which are basic components of structures. Furthermore, it has become important to understand nonlinear vibrations of composite materials, which have been widely used as structural members due to their excellent mechanical properties, as well as isotropic ones.

Chia [1,2] and Sathyamoorthy [3] have conducted a comprehensive review of the literature dealing with nonlinear problem in plates. There are also exhaustive literature reviews on nonlinear vibrations of shell-type structures due to Moussaoui and Benamar [4] and Amabili and Païdoussis [5]. As for studies on the internal

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resonance, which is one of typical nonlinear dynamics behaviors, of composite plates and shells, Hadian et al. [6] used the averaged Lagrangian and investigated two-to-one internal resonances of antisymmetric cross-ply laminated rectangular plates. Chaotic responses of a parametrically excited cross-ply laminated plate with a one-to-one internal resonance were examined by Ye et al. [7,8]. The present authors [9–11] analyzed multi-mode responses of simply supported laminated plates by applying Galerkin's procedure and the method of multiple scales (MMS), in which new detuning parameter was introduced to use the MMS properly. Primary resonances of antisymmetric mode for simply supported cross-ply laminated shells with the one-to-one internal resonance were investigated by the present authors [12], who reported that the addition of the fundamental vibration mode to the displacement functions overcame the shortcomings of the Galerkin discretization e.g. [13] for asymmetric vibration modes of continuous systems with quadratic and cubic nonlinearities.

This paper analyzes nonlinear dynamic responses of clamped laminated shallow shells with the internal resonance of $\omega_2 \approx \omega_3$, where ω_2 and ω_3 are natural frequencies of two asymmetric (second and third) vibration modes, by using the combination of Galerkin's procedure and the shooting method. First, we deal with the linear vibration problem of the shells, and calculate the eigenfunctions by the Ritz method. In the nonlinear vibration analysis, displacements of the shells are approximated by using the eigenfunctions of the first symmetric vibration mode in addition to ones of the two asymmetric vibration modes, and then the Galerkin discretization approach yields nonlinear ordinary differential equations. Finally, we apply the shooting method to the equations and obtain the frequency–response curves of the shells when a driving frequency Ω is near ω_2. In numerical examples, we treat with antisymmetric angle-ply shells and an isotropic shell, and show the nonlinear dynamic characteristics in the form of diagrams.

2. Equation of motion for a laminated shallow shell

We consider a laminated shallow shell of rectangular planform, which is composed of N orthotropic layers of uniform thickness, with lengths a and b, thickness h and radii of curvature R_x and R_y, as shown in Fig. 1. The coordinate system (x, y, z) is taken in the midsurface of the shell. The principal directions of elasticity are denoted by L and T, and θ_k is the angle between L and x axes in the kth layer. The components of the displacement at an arbitrary point of the shell in the x, y and z directions are u, v and w, respectively.

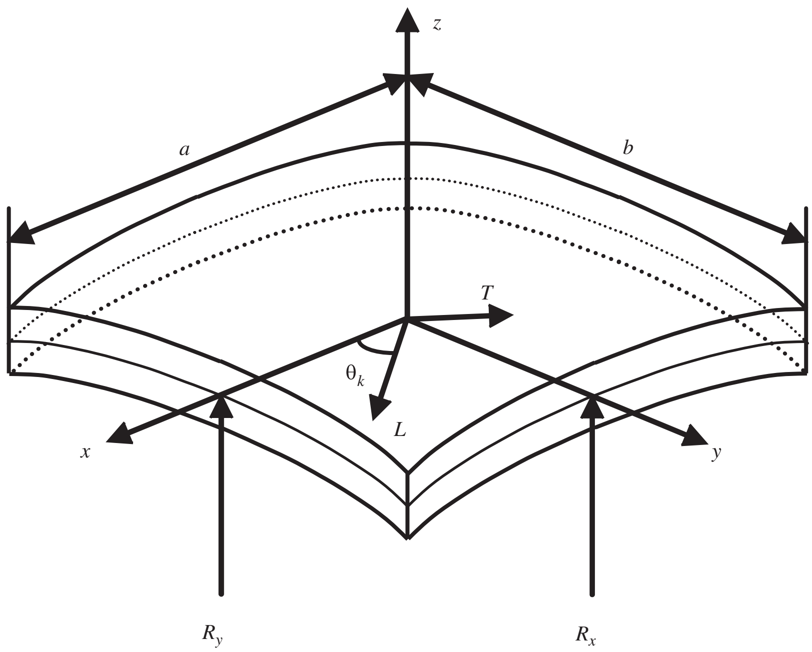

Fig. 1. Geometry of a laminated shallow shell and coordinate systems.

According to the first-order shear deformation theory, the in-plane displacements u and v are linear functions of the coordinate z, and the transverse displacement w is constant throughout the thickness of the shell. Under this assumption the displacement field may be given in the following form:

$$u = u_0 + z\psi_x, \quad v = v_0 + z\psi_y, \quad w = w_0, \tag{1}$$

where u_0, v_0 and w_0 are the displacements at the midsurface, ψ_x and ψ_y are the rotations of the midsurface about the y and x axes, respectively. The nonlinear strain–displacement relations of the shallow shell can be written as

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \varepsilon_z = 0, \quad \varepsilon_{xy} = \varepsilon_{xy}^0 + z\kappa_{xy}, \quad \varepsilon_{xz} = w_{0,x} - \frac{u_0}{R_x} + \psi_x, \quad \varepsilon_{yz} = w_{0,y} - \frac{v_0}{R_y} + \psi_y, \tag{2}$$

in which

$$\varepsilon_x^0 = u_{0,x} + \frac{w_0}{R_x} + \frac{w_{0,x}^2}{2}, \quad \varepsilon_y^0 = v_{0,y} + \frac{w_0}{R_y} + \frac{w_{0,y}^2}{2}, \quad \varepsilon_{xy}^0 = u_{0,y} + v_{0,x} + w_{0,x}w_{0,y}, \tag{3}$$

$$\kappa_x = \psi_{x,x}, \quad \kappa_y = \psi_{y,y}, \quad \kappa_{xy} = \psi_{x,y} + \psi_{y,x}, \tag{4}$$

and the subscripts following a comma stand for partial differentiation.

The stress, moment and shear stress resultants of the composite shallow shell can be expressed as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & \text{Sym.} & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{5}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} S_{44} & S_{45} \\ S_{45} & S_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}. \tag{6}$$

The elements A_{ij}, B_{ij}, D_{ij} and S_{ij} in the above equations are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & C_{16} \\ & C_{22} & 0 & 0 & C_{26} \\ & & C_{44} & C_{45} & 0 \\ & \text{Sym.} & & C_{55} & 0 \\ & & & & C_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix}, \tag{7}$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_k-1}^{h_k} C_{ij}^{(k)}(1, z, z^2)\,\mathrm{d}z, \quad i,j = 1, 2, 6, \tag{8}$$

$$S_{ij} = K^2 A_{ij} = K^2 \sum_{k=1}^{N} \int_{h_k-1}^{h_k} C_{ij}^{(k)}\,\mathrm{d}z, \quad i,j = 4, 5, \tag{9}$$

where the stiffness matrix elements ${C_{ij}}^{(\mathrm{k})}$ express the stress–strain relation in the kth layer, K^2 is the shear correction factor and h_k is the distance from the midsurface to the upper surface of the kth layer.

Considering the kinetic energy, the strain energy and the work done by an external pressure $q(x,y,t)$ acting in the z direction, using Hamilton's principle, and then the equations of motion are derived in non-dimensional form as [14]

$$U_{,\tau\tau} - \bar{N}_{\xi,\xi} - \alpha\bar{N}_{\xi\eta,\eta} - \frac{\bar{Q}_{\xi}}{2r_x} = 0, \tag{10}$$

$$V_{,\tau\tau} - \bar{N}_{\xi\eta,\xi} - \alpha\bar{N}_{\eta,\eta} - \frac{\bar{Q}_{\eta}}{2r_y} = 0, \tag{11}$$

$$W_{,\tau\tau} + \frac{\bar{N}_{\xi}}{2r_x} + \frac{\bar{N}_{\eta}}{2r_y} - (\bar{Q}_{\xi,\xi} + \alpha\bar{Q}_{\eta,\eta}) - 2H(\bar{N}_{\xi}W_{,\xi} + \alpha\bar{N}_{\xi\eta}W_{,\eta})_{,\xi} - 2H\alpha(\bar{N}_{\xi\eta}W_{,\xi} + \alpha\bar{N}_{\eta}W_{,\eta})_{,\eta} - q* = 0, \tag{12}$$

$$\frac{\psi_{x,\tau\tau}}{12} - \bar{M}_{\xi,\xi} - \alpha\bar{M}_{\xi\eta,\eta} - \frac{\bar{Q}_{\xi}}{2H} = 0, \tag{13}$$

$$\frac{\psi_{y,\tau\tau}}{12} - \bar{M}_{\xi\eta,\xi} - \alpha\bar{M}_{\eta,\eta} - \frac{\bar{Q}_{\eta}}{2H} = 0. \tag{14}$$

In the above equations, $\bar{N}$, $\bar{M}$ and $\bar{Q}$ are the non-dimensional stress, moment and shear stress resultants, respectively. Other non-dimensional parameters are defined as

$$\xi = \frac{2x}{a}, \quad \eta = \frac{2y}{b}, \quad U = \frac{u_0}{h}, \quad V = \frac{v_0}{h}, \quad W = \frac{w_0}{h}, \quad \alpha = \frac{a}{b}, \quad H = \frac{h}{a}, \quad r_x = \frac{R_x}{a}, \quad r_y = \frac{R_y}{a},$$
$$q* = \frac{a^3 q}{D_0 H}, \quad \tau = \frac{1}{a^2}\sqrt{\frac{D_0}{\rho h}}\, t, \tag{15}$$

with

$$D_0 = \frac{E_T h^3}{12(1 - v_{LT}v_{TL})}.$$

3. Free vibration analysis

In this section, we treat linear strain–displacement relations and obtain linear eigenfunctions of clamped laminated shallow shells.

In the linear vibration analysis, the maximum kinetic and strain energies of the shell are given as

$$T_{\max} = \frac{D_0 H^2 \lambda^2}{8\alpha} \int_{-1}^{1}\int_{-1}^{1} \left(U^2 + V^2 + W^2 + \frac{\psi_x^2 + \psi_y^2}{12} \right) \mathrm{d}\xi\, \mathrm{d}\eta, \tag{16}$$

$$U_{\max} = \frac{D_0 H^2}{8\alpha} \int_{-1}^{1}\int_{-1}^{1} \left(\begin{Bmatrix} \{\bar{\varepsilon}_l\} \\ \{\bar{\kappa}\} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} [A^*_{ij}] & [B^*_{ij}] \\ [B^*_{ij}] & [D^*_{ij}] \end{bmatrix} \begin{Bmatrix} \{\bar{\varepsilon}_l\} \\ \{\bar{\kappa}\} \end{Bmatrix} + \begin{Bmatrix} \bar{\varepsilon}_{\eta z} \\ \bar{\varepsilon}_{\xi z} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A^*_{44} & A^*_{45} \\ A^*_{45} & A^*_{55} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{\eta z} \\ \bar{\varepsilon}_{\xi z} \end{Bmatrix} \right) \mathrm{d}\xi\, \mathrm{d}\eta, \tag{17}$$

respectively, where

$$\{\bar{\varepsilon}_l\} = \{\bar{\varepsilon}_{\xi l}, \bar{\varepsilon}_{\eta l}, \bar{\varepsilon}_{\xi\eta l}\}^{\mathrm{T}}, \quad \{\bar{\kappa}\} = \{\bar{\kappa}_{\xi}, \bar{\kappa}_{\eta}, \bar{\kappa}_{\xi\eta}\}^{\mathrm{T}}, \quad \bar{\varepsilon}_{\xi l} = U_{,\xi} + \frac{W}{2r_x}, \quad \bar{\varepsilon}_{\eta l} = \alpha V_{,\eta} + \frac{W}{2r_y}, \quad \bar{\varepsilon}_{\xi\eta l} = \alpha U_{,\eta} + V_{,\xi},$$
$$\bar{\varepsilon}_{\xi z} = W_{,\xi} - \frac{U}{2r_x} + \frac{\psi_x}{2H}, \quad \bar{\varepsilon}_{\eta z} = \alpha W_{,\eta} - \frac{V}{2r_y} + \frac{\psi_y}{2H}, \bar{\kappa}_{\xi} = \psi_{x,\xi}, \quad \bar{\kappa}_{\eta} = \alpha\psi_{y,\eta}, \quad \bar{\kappa}_{\xi\eta} = \alpha\psi_{x,\eta} + \psi_{y,\xi},$$
$$A^*_{ij} = \frac{4a^2}{D_0} A_{ij}, \quad B^*_{ij} = \frac{4a^2}{D_0 h} B_{ij}, \quad D^*_{ij} = \frac{4a^2}{D_0 h^2} D_{ij}, \tag{18}$$

and λ is the non-dimensional natural frequency, which is related to the natural frequency $\bar{\lambda}$ by $\lambda = \bar{\lambda}a^2(\rho h/D_0)^{1/2}$. We consider that the shell is clamped along its four edges, and the boundary conditions are written as follows:

$$U = V = W = \psi_x = \psi_y = 0 \quad \text{at} \quad \xi = \pm 1 \quad \text{and} \quad \eta = \pm 1. \tag{19}$$

The displacements of the shell satisfying the boundary conditions (19) are expressed to be of the form [15,16]

$$U = \sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij}\phi_i(\xi)\phi_j(\eta), \quad V = \sum_{i=1}^{I}\sum_{j=1}^{J} b_{ij}\phi_i(\xi)\phi_j(\eta), \quad W = \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}\phi_i(\xi)\phi_j(\eta),$$
$$\psi_x = \sum_{i=1}^{I}\sum_{j=1}^{J} d_{ij}\phi_i(\xi)\phi_j(\eta), \quad \psi_y = \sum_{i=1}^{I}\sum_{j=1}^{J} e_{ij}\phi_i(\xi)\phi_j(\eta), \quad \phi_i(\xi) = \xi^{i-1}(1-\xi^2), \quad \phi_j(\eta) = \eta^{i-1}(1-\eta^2), \tag{20}$$

where a_{ij}, b_{ij}, c_{ij}, d_{ij} and e_{ij} are unknown coefficients. By substituting the equations, which are obtained by the substitution of Eq. (20) into Eqs. (16) and (17), into the conditions for a stationary value of the Lagrange functional $L_a = T_{\max} - U_{\max}$,

$$\frac{\partial L_a}{\partial a_{ij}} = \frac{\partial L_a}{\partial b_{ij}} = \frac{\partial L_a}{\partial c_{ij}} = \frac{\partial L_a}{\partial d_{ij}} = \frac{\partial L_a}{\partial e_{ij}} = 0 \tag{21}$$

a frequency equation is derived. These unknown coefficients are obtained as the eigenfunctions by solving the eigenvalue problem.

4. Forced vibration analysis

In the present paper, we consider an internal resonance when natural frequencies ω_2 and ω_3 of two asymmetric (second and third) vibration modes have the relationship $\omega_2 \approx \omega_3$. On the same problem of flat plates, the displacements are approximated by using the linear eigenfunction of the two vibration modes, and then the governing equations are discretized by using Galerkin's procedure e.g. [10]. However, in nonlinear vibration analyses for asymmetric modes of shells, it was reported that displacements should be approximated by using a fundamental vibration (first symmetric vibration) mode in addition to the asymmetric modes in order to capture properly the nonlinear dynamics through Galerkin's procedure [12,14]. Thus, the displacements of the shell can be expressed using the eigenfunctions of the first symmetric vibration mode in addition to ones of the two asymmetric vibration modes, which are calculated by the Ritz method, as

$$U = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(U_1(\tau)a_{ij}^{(1)} + U_2(\tau)a_{ij}^{(2)} + U_3(\tau)a_{ij}^{(3)}\right)\phi_i(\xi)\phi_j(\eta),$$
$$V = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(V_1(\tau)b_{ij}^{(1)} + V_2(\tau)b_{ij}^{(2)} + V_3(\tau)b_{ij}^{(3)}\right)\phi_i(\xi)\phi_j(\eta),$$
$$W = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(W_1(\tau)c_{ij}^{(1)} + W_2(\tau)c_{ij}^{(2)} + W_3(\tau)c_{ij}^{(3)}\right)\phi_i(\xi)\phi_j(\eta),$$
$$\psi_x = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(X_1(\tau)d_{ij}^{(1)} + X_2(\tau)d_{ij}^{(2)} + X_3(\tau)d_{ij}^{(3)}\right)\phi_i(\xi)\phi_j(\eta),$$
$$\psi_y = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(Y_1(\tau)e_{ij}^{(1)} + Y_2(\tau)e_{ij}^{(2)} + Y_3(\tau)e_{ij}^{(3)}\right)\phi_i(\xi)\phi_j(\eta), \tag{22}$$

in which time functions $U(\tau)$, $V(\tau)$, $W(\tau)$, $X(\tau)$ and $Y(\tau)$ with subscripts 1, 2 and 3 are amplitudes of the first, second and third modes, respectively. In a similar way, coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij} and e_{ij} with superscripts (1), (2) and (3) stand for eigenvectors of the first, second and third modes, respectively.

We assume that only the second vibration mode is directly excited by the harmonic load, and its distribution is defined as

$$q* = F\sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}^{(2)}\phi_i(\xi)\phi_j(\eta)\cos\Omega\tau, \tag{23}$$

where F and Ω are the non-dimensional amplitude and angular frequency of the load, respectively. By substituting Eqs. (22) and (23) into the equations of motion (10)–(14) and applying Galerkin's procedure,

$$\int_{-1}^{1}\int_{-1}^{1} F_U \sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij}^{(l)}\phi_i(\xi)\phi_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \int_{-1}^{1}\int_{-1}^{1} F_V \sum_{i=1}^{I}\sum_{j=1}^{J} b_{ij}^{(l)}\phi_i(\xi)\phi_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0,$$
$$\int_{-1}^{1}\int_{-1}^{1} F_W \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}^{(l)}\phi_i(\xi)\phi_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \int_{-1}^{1}\int_{-1}^{1} F_X \sum_{i=1}^{I}\sum_{j=1}^{J} d_{ij}^{(l)}\phi_i(\xi)\phi_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0,$$
$$\int_{-1}^{1}\int_{-1}^{1} F_Y \sum_{i=1}^{I}\sum_{j=1}^{J} e_{ij}^{(l)}\phi_i(\xi)\phi_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad (l = 1, 2, 3) \tag{24}$$

and then 15 sets of simultaneous nonlinear differential equations are derived. In the above equations, we define F_U, F_V, F_W, F_X and F_Y as the equations obtained by substituting Eqs. (22) and (23) into the equations of motion (10), (11), (12), (13) and (14), respectively. If in-plane and rotatory inertias in these equations are neglected, three sets of ordinary differential equations in terms of the transverse displacements W_1, W_2 and W_3 are derived by eliminating U_i, V_i, X_i and Y_i and adding on the effect of viscous damping:

$$W_{1,\tau\tau} + \mu\omega_1 W_{1,\tau} + \omega_1^2 W_1 + G_1 W_1^2 + G_2 W_2^2 + G_3 W_3^2 + G_4 W_1^3 + G_5 W_1 W_2^2 + G_6 W_1 W_3^2 = 0,$$
$$W_{2,\tau\tau} + \mu\omega_2 W_{2,\tau} + \omega_2^2 W_2 + G_7 W_1 W_2 + G_8 W_1^2 W_2 + G_9 W_2^3 + G_{10} W_2 W_3^2 = F\cos\Omega\tau,$$
$$W_{3,\tau\tau} + \mu\omega_3 W_{3,\tau} + \omega_3^2 W_3 + G_{11} W_1 W_3 + G_{12} W_1^2 W_3 + G_{13} W_2^2 W_3 + G_{14} W_3^3 = 0, \tag{25}$$

where ω_i and G_i are the non-dimensional natural frequencies and coefficients of the nonlinear terms, respectively. They are calculated numerically by the use of the software package Mathematica [17]. The terms $\mu\omega_i W_{i,\tau}$ indicate the modal damping and μ is the non-dimensional damping ratio. The difference between two natural frequencies λ in Eq. (16) and ω in Eq. (25) is that the former is obtained by considering the effects of in-plane and rotatory inertias (the Ritz method), whereas the latter is done by neglecting the effects (Galerkin's procedure). The terms $\mu\omega_i W_{i,\tau}$ are additional modal damping. In the above equation, by considering the effect of the first vibration mode W_1, the quadratic nonlinear terms $G_7 W_1 W_2$ and $G_{11} W_1 W_3$ appear in the equation expressing the asymmetric modes. For example, when W_2 is excited, W_1 is always activated by the nonlinear $G_2 W_2^2$, and then W_1 affects W_2 through the nonlinear terms $G_7 W_1 W_2$ and $G_8 W_1^2 W_2$ (i.e., the coupled vibration between W_1 and W_2 occurs). This modal interaction is specific to shell [12,14]. In order to capture the nonlinear dynamic characteristics properly, the fundamental mode should be considered in the analysis.

The steady-state responses of the shells under the conditions ($\Omega \approx \omega_2$, $\omega_2 \approx \omega_3$) are obtained by applying the shooting method, which is calculating not only the stable solutions but also unstable ones, to Eq. (25). The algorithm of the shooting method is well described in Refs. [12,18].

5. Numerical results and discussion

In this section, the nonlinear dynamic behaviors of the shells with the one-to-one internal resonance ($\omega_2 \approx \omega_3$) when Ω is near ω_2 are presented in the form of diagrams. Here, we treat antisymmetric angle-ply laminated shells ($\theta = 45°/-45°/45°/-45°$) which consists of graphite-epoxy layers and an isotropic shell whose

Table 1
Frequency parameters ω_i of shells treated in numerical examples

	θ	r_x	H	ω_1	ω_2	ω_3
Case I	45°/−45°/45°/−45°	10	0.01	142.6	191.4	192.1
Case II	45°/−45°/45°/−45°	10	0.05	81.52	151.7	151.9
Case III	45°/−45°/45°/−45°	25	0.01	96.32	173.2	173.5
Case IV	Isotropic	10	0.01	58.27	81.65	81.65

Poisson's ratio $v = 0.3$. The material properties of graphite-epoxy composite are

$$E_L = 138\,\text{GPa}, \quad E_T = 8.96\,\text{GPa}, \quad G_{LT} = 7.1\,\text{GPa}, \quad G_{TZ} = E_T/2, \quad v_{LT} = 0.3.$$

In the following numerical examples, the shear correction factor K^2, the damping ratio μ and the amplitude F of the load are taken as $K^2 = 5/6$, $\mu = 0.01$ and $F/\omega_2^2 = 0.01$, respectively. The non-dimensional natural frequencies ω_i of the shells presented in this section are listed in Table 1.

In the present analysis, the accuracy of numerical results is dependent upon the number of series (20). We hence examine first of all convergence characteristics of steady-state responses as the number of terms I and J of Eq. (20) increases. Fig. 2 shows a convergence study for the stable three-mode response induced by the internal resonance (i.e., $W_3 \neq 0$) of the shell (Case I), where note that the maximum amplitude of W_1 is negative value [12]. It is clearly seen that the responses converge with an increase in number of the series, and the result by using $I \times J = 8 \times 8$ exactly coincides with that by using $I \times J = 10 \times 10$. Therefore, $I \times J = 8 \times 8$ is adopted in the present calculations.

Frequency–response curves for the shell (Case I) is shown in Fig. 3, in which the solid and broken lines denote stable and unstable responses, respectively. It is seen from Fig. 3 that a stable three-mode response ($W_3 \neq 0$), which is activated the internal resonance, occurs at $\Omega/\omega_2 \approx 1.000$ via a pitchfork bifurcation, and then the coupled response between the first and second vibration mode loses its stability. As the driving frequency is increased from low frequency (e.g., $\Omega/\omega_2 = 0.98$), the coupled response changes into a three-mode response through the bifurcation point continuously. The coupled response, which loses its stability via the bifurcation point, recovers its stability at $\Omega/\omega_2 \approx 1.009$ via a pitchfork bifurcation. There exists three stable responses in the region given by $1.035 \leqslant \Omega/\omega_2 \leqslant 1.056$, one of them may generate by the initial condition. Furthermore, there is also a stable three-mode response, which occurs via saddle-node bifurcation and loses its stability via a Hopf bifurcation in very small region at $\Omega/\omega_2 \approx 1.029$.

Fig. 4 depicts the effect of radius of curvature and thickness ratio on the three-mode responses of a spherical laminated shallow shell. Only stable three-mode responses are plotted in the figure, where solid, broken and dotted lines denote the results for the Case I, II and III, respectively. We confirmed that the three-mode responses occurred and vanished via the same process as shown in Fig. 3. Although there is also the three-mode response generated via a saddle-node bifurcation, the responses omitted here because its region is very narrow. It is found from Fig. 4 that the range of the three-mode responses becomes wider with increases of radius of curvature and thickness ratio.

Fig. 5 presents the frequency–response curves for the spherical isotropic shell (Case IV). There exist two stable three-mode, which occur via a pitchfork bifurcation and via a saddle-node bifurcation, in the isotropic shell. As seen in Figs. 3 and 5, this tendency is the same as the laminated shell. However, the unstable coupled response passing the pitchfork bifurcation point ($\Omega/\omega_2 \approx 1.000$) does not change the stability. Further, in the case of the isotropic shell, the range of the stable three-mode response is narrower than that of the laminated shell.

6. Concluding remarks

Nonlinear responses of clamped laminated shallow shells with a one-to-one internal resonance between two antisymmetric modes were investigated in the present paper. We derived nonlinear ordinary differential

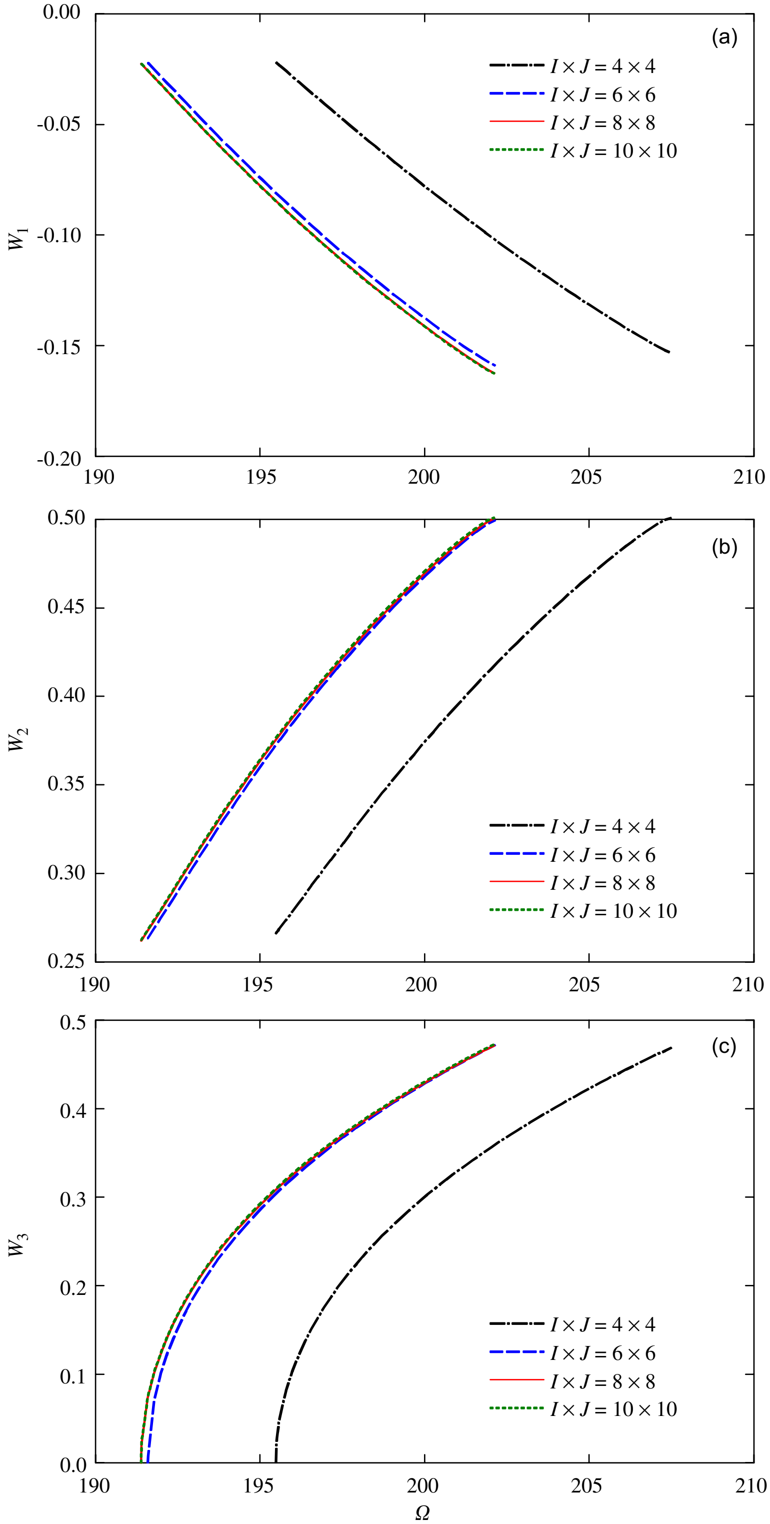

Fig. 2. Convergence characteristics of frequency–response curves for the laminated shell (Case I, $W_3 \neq 0$): (a) first mode, (b) second mode, and (c) third mode.

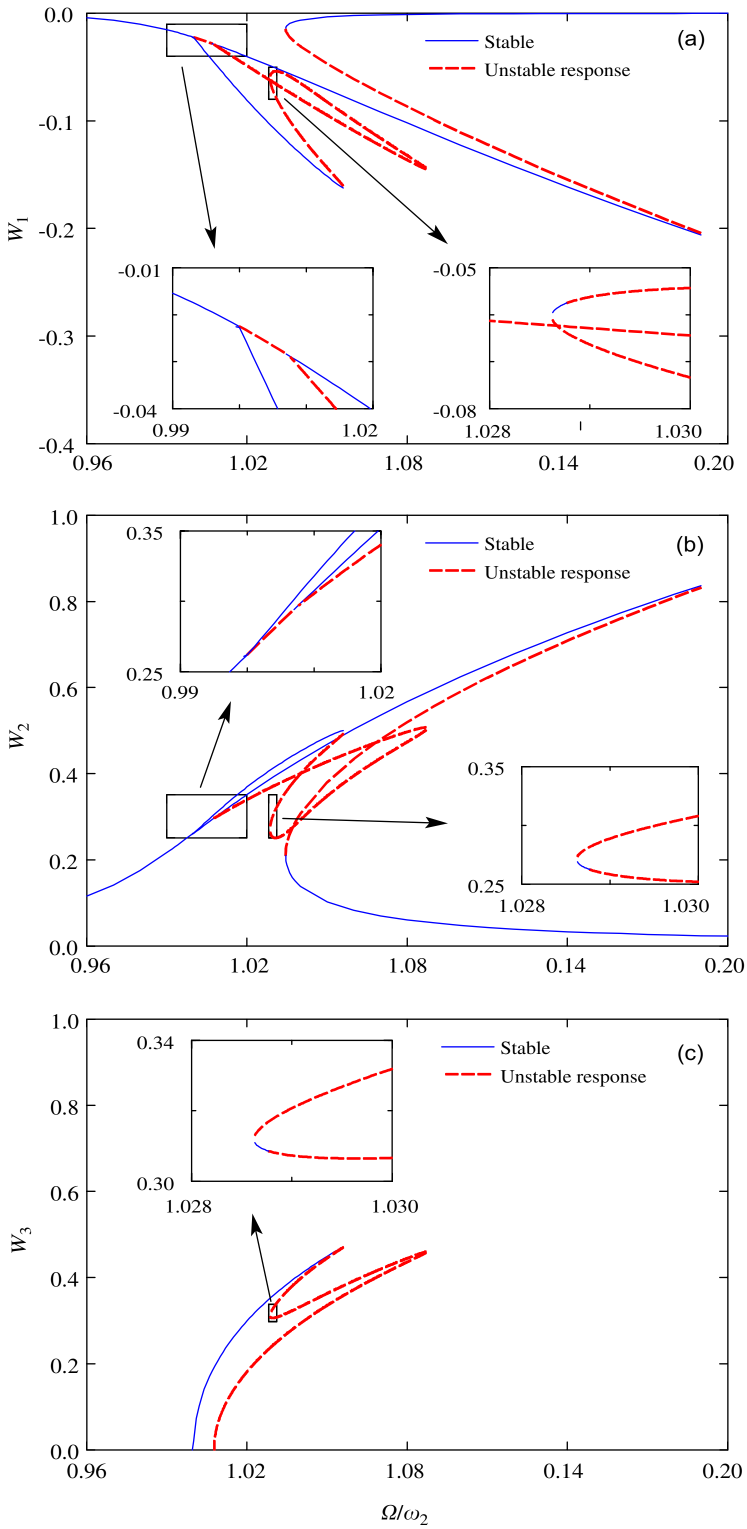

Fig. 3. Frequency–response curves for the laminated shell (Case I): (a) first mode, (b) second mode, and (c) third mode.

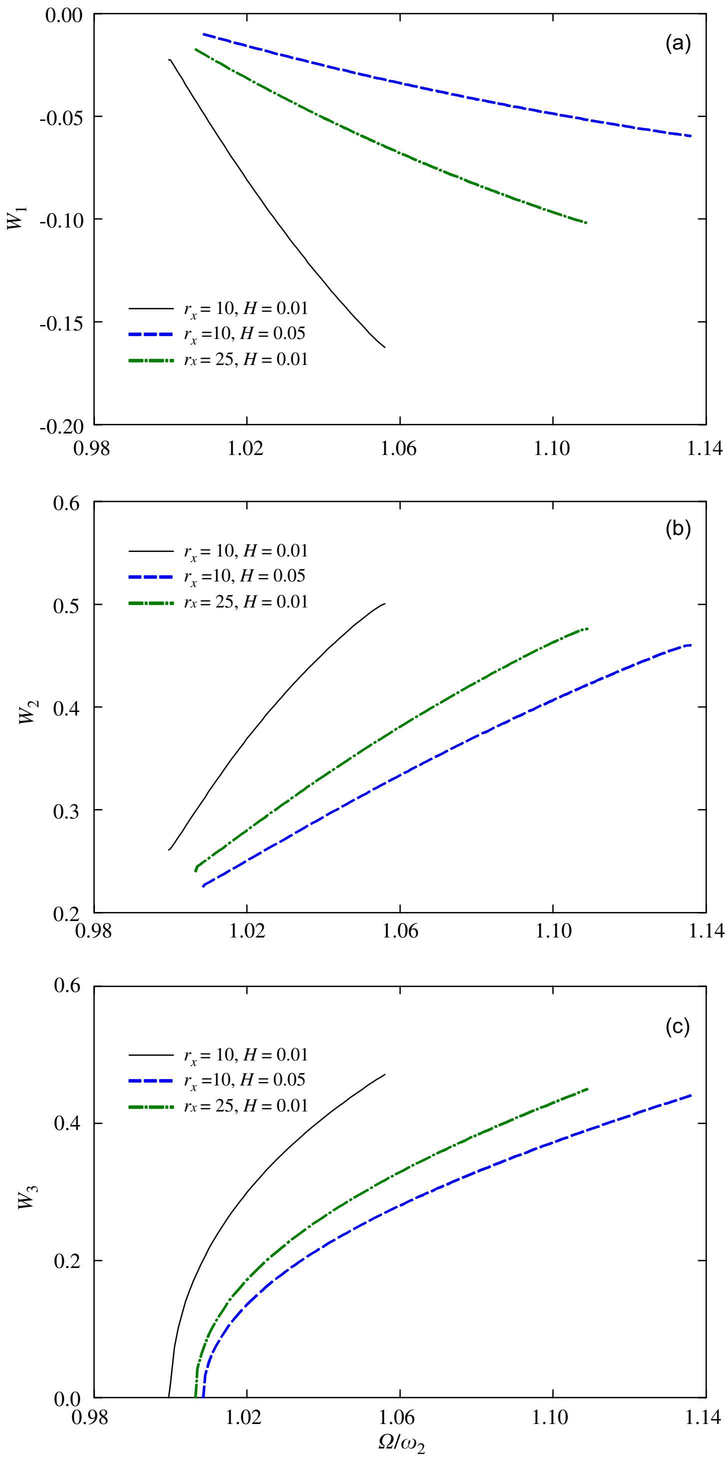

Fig. 4. Effects of curvature and thickness ratios on frequency–response curves for the laminated shell ($W_3 \neq 0$): (a) first mode, (b) second mode, and (c) third mode.

equations by applying Galerkin discretization approach to the equations of motion, in which the displacements of the shells were approximated by the symmetric first vibration mode in addition to the two asymmetric vibration modes to capture the nonlinear dynamic characteristics properly. In order to examine

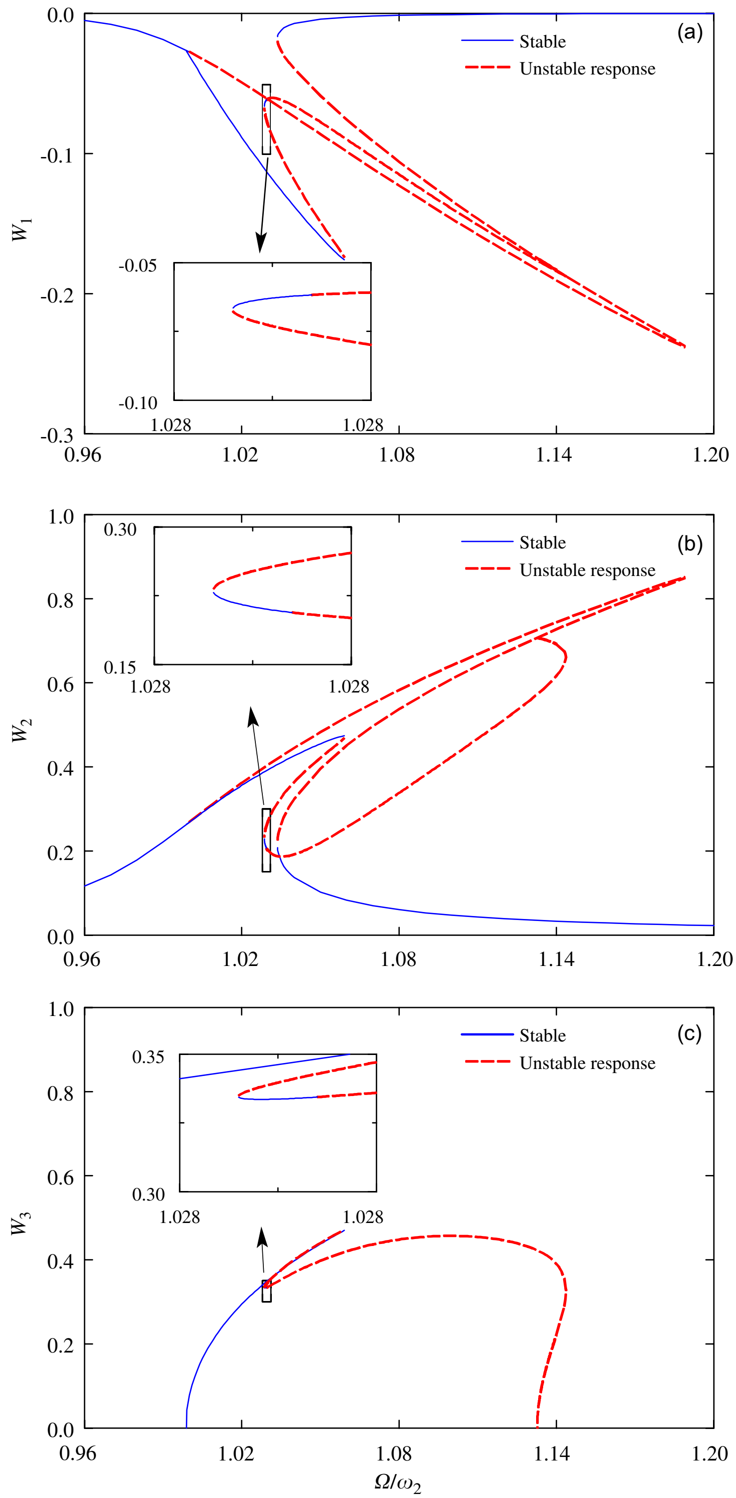

Fig. 5. Frequency–response curves for the isotropic shell (Case IV): (a) first mode, (b) second mode, and (c) third mode.

the dynamic behaviors when a driving frequency Ω is near the natural frequency ω_2 of the second vibration mode, the frequency–response curves were obtained by the shooting method. In numerical examples, the influence of radius of curvature and thickness ratio on the responses activated by the internal resonance was shown. Further, we also treated with an isotropic shell, and then the difference between isotropic and composite shells was made clear in the form of diagrams.

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